EMA 3702
Mechanics & Materials Science
(Mechanics of Materials)

Chapter 3  Torsion
Introduction

Stress and strain in components subjected to torque $T$

Cross-section shape

- Circular
  - Non-circular
  - Irregular shape

Material

- Elastic
  - Elastoplastic

Shaft design

- Solid
  - Hollow
Introduction

Component subject to twisting couples, or torques of $T & T'$

- $T$ is a vector and has two ways of representations

Curved arrow & Couple vector (right hand rule)

Example:
Transmission of torque in shafts
Stresses in a Shaft under Torsion (1)

A shaft subjected to equal and opposite torques (or moments of force) of $T$ and $T'$ at $A$ and $B$

In a normal cross-section at $C$, for an elementary area $dA$, the element shearing force $dF$ and local stress $\tau$ satisfy:

$$dF = \tau dA$$

Define $\rho$ as (local) lever arm, i.e., the perpendicular distance from the elemental area to the axis (center), total torque $T$ is:

$$\int \rho dF = T$$
Stresses in a Shaft under Torsion (2)

From previous

\[ dF = \tau dA \]

\[ \int \rho dF = T \]

Therefore,

\[ \int \rho (\tau dA) = T \]

Complications for torsion:
• Distributions of \( \tau \) and the resulting \( \gamma \), i.e., how they change over the cross-section plane is statically indeterminate.
• Unlike normal stress or simple shearing, distribution of \( \tau \) for torsion is **NOT uniform**!
Axial Shearing Stress in Shaft under Torsion

Consider a small element as illustrated. Based on previous considerations for shearing stress, if \( \tau_{xz} \neq 0 \), axial shearing stress

\[
\tau_{zx} = \tau_{xz} \neq 0
\]

Implication: under torsion, shearing stress exists along longitude planes, and neighboring elements have tendency to slide against each other along axial direction!
Axi-symmetric Property of a Circular Shaft

For circular shaft under torsion

- Cross-sections remain **planar**
- Cross-sections remain **undistorted**
Shearing Strain in Circular Shaft under Torsion

Within each cross-section, NO change

Along axial direction, there is strain (deformation) due to axial shear $\tau_{zx}$

Define the following terms

$L$  Length along shaft axis

$\rho$  Radial distance from the shaft axis

$\phi$  Angle of twist for a cross-section

$\gamma$  Shearing strain (change in angle from $90^\circ$)

Far away from location of loading & for **small** strain $\gamma$ and angle $\phi$

$$\gamma L = \rho \phi$$

$$\gamma = \frac{\rho \phi}{L}$$
Shearing Strain in Circular Shaft under Torsion

\[ \gamma = \frac{\rho \phi}{L} \]

For a given \( L \), when \( \rho = c \), i.e., radius of the shaft

\[ \gamma_{\text{max}} = \frac{c \phi}{L} \]

Additionally, for a given \( L \)

\[ \frac{\gamma}{\gamma_{\text{max}}} = \frac{\rho}{c} \]

\[ \gamma = \frac{\rho}{c} \gamma_{\text{max}} \]
Shearing Stresses in Circular Shaft under Torsion

Hooke’s Law for shear

\[ \tau = G\gamma \]

\[ \gamma = \frac{\rho\phi}{L} \]

Therefore,

\[ \tau = \frac{G\rho\phi}{L} \]

Use when shaft twist angle, length, AND materials \( G \) are known

For given \( L \) and \( \phi \), when \( \rho = c = \) radius of the shaft, shearing stress reaches maximum:

\[ \tau_{\text{max}} = G\gamma_{\text{max}} = \frac{Gc\phi}{L} \]

The ratio between \( \tau \) and \( \tau_{\text{max}} \)

\[ \frac{\tau}{\tau_{\text{max}}} = \frac{\rho}{c} \]

Distribution of shearing stress \( \tau \) is **linear** w.r.t. radius from the axis \( \rho \)
Shearing Stresses & Torque in Circular Shaft under Torsion

Recall relationship of $T$ and local shearing stress $\tau$

$$\tau = \frac{\rho}{c} \tau_{\text{max}}$$

Recall the definition for moment of inertia: $J = \int \rho^2 \, dA$

For a circular shaft with fixed radius $c$ under torque $T$

$$T = \int \rho (\tau \, dA) = \int \rho \frac{\rho}{c} \tau_{\text{max}} \, dA = \frac{\tau_{\text{max}}}{c} \int \rho^2 \, dA$$

We have $T = \frac{\tau_{\text{max}} \, J}{c}$

Or $\tau_{\text{max}} = \frac{Tc}{J}$

and $\tau = \frac{T \rho}{J}$

Use when shaft geometry AND applied torque known
Solid & Hollow Circular Shaft under Torsion

For a solid cylinder

\[ \tau = \frac{T\rho}{J} \quad \tau_{\text{max}} = \frac{Tc}{J} \]

\[ J = \frac{1}{2} \pi c^4 \]

For a hollow cylinder

\[ J = \frac{1}{2} \pi (c_2^4 - c_1^4) \]

\[ \tau = \frac{T\rho}{J} \quad \frac{\tau_{\text{min}}}{\tau_{\text{max}}} = \frac{c_1}{c_2} \]
Example Problem for Torsion 3.1

Shaft $ABCD$ subject to torques as illustrated. Knowing section $BC$ is hollow with ID = 90 mm and OD = 120 mm. Sections $AB$ and $CD$ are solid. Calculate

(a) The maximum and minimum shearing stress in section $BC$;
(b) The required minimal diameter for $AB$ and $CD$ if shearing stress should not exceed 65 MPa.
Example Problem for Torsion 3.1

Section $AB$, $T_A = 6 \text{kN} \cdot \text{m}$

Net internal torque $T_{AB} = 6 \text{kN} \cdot \text{m}$

Section $BC$, $T_A = 6 \text{kN} \cdot \text{m}$

Net internal torque $T_{BC} = (6 + 14) \text{kN} \cdot \text{m} = 20 \text{kN} \cdot \text{m}$
Example Problem for Torsion 3.1

For section $BC$, $T_{BC} = 20 \text{ kN} \cdot \text{m}$

“section $BC$ is hollow with ID = 90 mm and OD = 120 mm.”

**Max stress in $BC$** occurs at the outer surface

$$
\tau_{\text{max}} = \frac{T_{BC} c_{\text{outer}}}{J_{BC}} = \frac{20 \times 10^3 N \cdot m \times 0.06m}{1 \times 3.1416 \times \left(0.06^4 - 0.045^4\right)m^4} = 86.2 \text{ MPa}
$$

**Min stress in $BC$** occurs at the inner surface

$$
\tau_{\text{min}} = \frac{T_{BC} c_{\text{inner}}}{J_{BC}} = \frac{20 \times 10^3 N \cdot m \times 0.045m}{1 \times 3.1416 \times \left(0.06^4 - 0.045^4\right)m^4} = 64.7 \text{ MPa}
$$
Example Problem for Torsion 3.1

Section AB, $T_{AB} = 6$ kN·m
Section BC, $T_{BC} = 20$ kN·m
Section CD, $T_{CD} = 6$ kN·m

For both AB and CD, “shearing stress should not exceed 65 MPa”

$$
\tau_{\text{max}} = \frac{T}{J} = \frac{T}{\frac{1}{2} \pi c^3} \leq 65 \text{MPa}
$$

Minimum radius for AB or CD:

$$
c \geq \left( \frac{T}{\frac{1}{2} \pi \tau_{\text{max}}} \right)^{1/3} = \left( \frac{6 \times 10^3 \text{ N} \cdot \text{m}}{\frac{1}{2} \times 3.1416 \times 65 \times 10^6 \text{ N} / \text{m}^2} \right)^{1/3} = 3.89 \text{ cm}
$$

Minimum diameter = 7.78 cm
A hollow cylinder shaft is 1 m long and has inner and outer diameter of 20 and 40 mm. (a) What is the largest torque that can be applied if shearing stress should not exceed 100 MPa? (b) What is the minimum shearing stress when maximum reaching 100 MPa?

From geometry, moment of inertia

\[ J = \frac{1}{2} \pi \left( c_{\text{outer}}^4 - c_{\text{inner}}^4 \right) = \frac{1}{2} \times 3.1416 \times \left( 0.02^4 - 0.01^4 \right) = 2.35 \times 10^{-7} \text{ m}^4 \]

“shearing stress should not exceed 100 MPa”:

\[ \tau_{\text{max}} = \frac{Tc_{\text{outer}}}{J} \leq 100 \text{ MPa} \]

Largest torque that can be applied:

\[ T \leq \frac{J \cdot 100 \text{ MPa}}{c_{\text{out}}} = \frac{2.35 \times 10^{-7} \text{ m}^4 \times 100 \times 10^6 \text{ N/m}^2}{0.02 \text{ m}} = 1175 \text{ N.m} \]

Min shearing stress when max is 100 MPa

\[ \tau_{\text{min}} = \frac{c_{\text{inner}}}{c_{\text{outer}}} \tau_{\text{max}} = \frac{0.01}{0.02} \times 100 \text{ MPa} = 50 \text{ MPa} \]
Normal Stress in Circular Shaft under Torsion

For element \( a \) at shaft surface

\[
\tau_{\text{max}} = \frac{Tc}{J}
\]

For half of \( a \) at \( 45^\circ \), shear force along \( BC \) and \( BD \) are:

\[
F_{BC} = F_{BD} = \tau_{\text{max}} A_0
\]

To balance, force on \( CD \) surface \( F_{CD} \) must be normal force

\[
F_{CD} = F_{BC} / \cos 45^\circ = \sqrt{2}\tau_{\text{max}} A_0
\]

Since \( A_{CD} = \sqrt{2}A_0 \)

Normal stress due to torsion

\[
\sigma = \frac{F_{CD}}{A_{CD}} = \tau_{\text{max}} = \frac{Tc}{J}
\]

For torsion, significant normal stress still exists and may cause failure!
Failure of Material under Torsion

For circular shaft under torsion

- **Ductile** materials are weaker in shear and fail with $90^\circ$ fracture (i.e., fracture surface perpendicular to axis)
- **Brittle** materials are weaker in tension and fail with $45^\circ$ fracture (i.e., fracture surface at $45^\circ$ from axis)

Photo by Jeff Thomas, 11/1997
http://classes.mst.edu/civeng120/lessons/torsion/fracture/index.html
Angle of Twist within Elastic Range

For torsion, based on geometry

\[ \gamma_{\text{max}} = \frac{c\phi}{L} \]

Within elastic limit, Hooke’s law

\[ \gamma_{\text{max}} = \frac{\tau_{\text{max}}}{G} \]

Max sharing stress due to torque

\[ \tau_{\text{max}} = \frac{Tc}{J} \]

\[ \frac{\phi}{L} = \frac{T}{JG} \]

\[ \phi = \frac{TL}{JG} \]

Angle of twist \( \phi \) increases with
- Increasing \( T \) and \( L \)
- Decreasing \( J \) and \( G \)
Angle of Twist for Multiple Cross-Section Shafts

Overall twist of A w.r.t B

\[ \phi = \sum_i \frac{T_i L_i}{J_i G_i} \]

in which \( T_i, J_i, \) and \( L_i, G_i \) are obtained and/or analyzed for each section

Angle of Twist for Variable Cross-Section Shafts

For element \( dx \)

\[ d\phi = \frac{Tdx}{JG} \]

Overall

\[ \phi = \int_0^L \frac{Tdx}{JG} \]
A cylindrical shaft is 0.5 m long and has diameter of 20 mm fixed at one end. If a torque of 76 N·m is applied to its free end, and the measured angle of twist $\phi$ at the loading end is 1.8°, please estimate the shear modulus knowing moment of inertia is

$$J = \frac{1}{2} \pi c^4$$

Angle of twist $\phi = \frac{TL}{JG}$

Shear modulus $G$

$$G = \frac{TL}{J\phi} = \frac{76N \cdot m \times 0.5m}{\frac{1}{2} \times 3.1416 \times (0.01)^4 m^4 \times \frac{1.8}{180} \times 3.1416} = 77GPa$$
A cylindrical shaft $AB$ with two sections is fixed between two rigid end support at $A$ and $B$. Section $AC$ is solid with diameter of 1.0 in; Section $CB$ is hollow with outside diameter of 1.0 in and inner diameter of 0.8 in. Both sections have length of 10 in. If an external torque of 100 lb·ft is applied at center $C$, please determine the reaction torque at $A$ and $B$, assuming the deformation is within proportional limit.
Example of Statically Indeterminate Shaft (2)

From statics, balance of torque

$$T = T_A + T_B = 100\text{lb} \cdot \text{in}$$

One equation – two variables

→ **Statically indeterminate**

Consider geometry/deformation:

For section AC, angle of twist $\phi_{AC}$

For section CB, angle of twist $\phi_{CB}$

$$\phi_{AC} = \phi_{CB}$$

Therefore,

$$\phi_{AC} = \frac{T_A L_{AC}}{J_{AC}G} = \frac{T_B L_{CB}}{J_{CB}G} = \phi_{CB}$$

Two equations & two variables, problem could be solved
Example of Statically Indeterminate Shaft (3)

Therefore,

\[ T_A + T_B = 100 \]

\[ \frac{T_A}{\frac{1}{2} \pi 0.5^4} = \frac{T_B}{\frac{1}{2} \pi (0.5^4 - 0.4^4)} \]

\[ \frac{T_A}{625} = \frac{T_B}{369} \]

Solving these two

\[ T_A = 62.9 lb \cdot ft \]

\[ T_B = 37.1 lb \cdot ft \]
Design of Transmission Shafts (1)

Two parameters in transmission shaft design:

• Power $P$
• Speed of rotation

For power: $P = T \omega$

$\omega$ is angular velocity, $\omega = 2\pi f$, and has unit of radians/s

$f$ is frequency in unit Hz or s$^{-1}$

Therefore, power

$$P = 2\pi f T$$

has unit of N·m/s = W

As a result, for given $P$ and $f$, the resulting torque will be

$$T = \frac{P}{2\pi f}$$
Design of Transmission Shafts (2)

Relationship between power output $P$, angular speed (or frequency $f$), and torque $T$

$$ T = \frac{P}{2\pi f} $$

On the other hand, max shearing stress for a given shaft geometry and applied torque $J$

$$ \tau_{\text{max}} = \frac{Tc}{J} \quad \Rightarrow \quad \frac{J}{c} = \frac{T}{\tau_{\text{max}}} $$

Example of a solid circular shaft, moment of inertia:

$$ J = \frac{1}{2} \pi c^4 \quad \Rightarrow \quad \frac{J}{c} = \frac{1}{2} \pi c^3 = \frac{T}{\tau_{\text{max}}} $$

$$ c \geq \left( \frac{2T}{\pi \tau_{\text{max\_allowable}}} \right)^{1/3} $$
Class Exercise

What size of solid shaft should be used for motor of 592 hp (1 hp = 6600 lb·in/s) operating at 3600 rpm \( (f = 60 \text{ Hz}) \) if the shearing stress is not to exceed 6600 psi in the shaft?

Power for the shaft \( P = 2\pi f T \)

Torque for the shaft \( T = \frac{P}{2\pi f} = \frac{592 \times 6600 \text{lb} \cdot \text{in} \cdot \text{s}^{-1}}{2 \times 3.1416 \times 60 \text{s}^{-1}} = 10382 \text{lb} \cdot \text{in} \)

Maximum shearing stress

\[
\tau_{\text{max}} = \frac{T_c}{J} = \frac{T_c}{1} = \frac{2T}{\pi c^4} \leq 6600 \text{lb} / \text{in}^2
\]

Minimum shaft radius:

\[
c \geq \left( \frac{2T}{\pi \tau_{\text{max, allowable}}} \right)^{1/3} = \left( \frac{2 \times 10382 \text{lb} \cdot \text{in}}{3.1416 \times 6600 \text{lb} / \text{in}^2} \right)^{1/3} = 1.00 \text{ in}
\]
Stress Concentrations in Circular Shafts

Similar to normal stress, shearing stress may concentrate (i.e., show higher value) at certain locations.

Concentration factor $K$

$$K = \frac{\tau_{\text{max\_actual}}}{Tc / J}$$

or

$$\tau_{\text{max\_actual}} = K \frac{Tc}{J}$$

Local max stress increases with

- Sharper transition or joining
- Larger ratio of radius
Plastic Deformation in Circular Shafts (1)

For a given shaft construction (geometry and material), when torque is low and within linear elastic region, linear stress distribution w.r.t. radius from axis

\[ \tau = \frac{T \rho}{J} \propto T \]

As torque increases further, shearing stress would reach yield strength \( \tau_Y \), it will go into non-linear (e.g., ideal elasoplastic) distribution of \( \tau \) w.r.t. radial from axis in certain region
Plastic Deformation in Circular Shafts (2)

For angle of twist $\phi$
When torque/shearing stress is low and within linear elastic region, $\phi$ increases linearly with torque $T$

$$\phi = \frac{TL}{JG} \propto T$$

As torque increases further, shearing stress reaches yield strength $\tau_Y$, it will go into non-linear distribution of $\phi$ w.r.t. $T$
Torsion of Noncircular Members

A rectangular or square shaft does **not** contain axi-symmetry

- The cross-sections generally do NOT remain flat or planar under torsion
- Shearing stress at the edge (corner) of the shaft is zero
- Maximum shearing stress occur in the middle of flat face
Thin-Walled Hollow Shafts

For arbitrary shaped thin walled hollow tube subjected to torque $T$, the shearing stress can be approximated as

$$\tau = \frac{T}{2ta}$$
Homework 3.0

Read chapter 3 textbook sections 3.1 to 3.5 and give an honor statement confirm reading.
Homework 3.1

Electric motor applies a torque of 2.8 kN\(\cdot\)m on shaft \(EF\) at \(E\). Knowing each shaft section is solid. Based on the additional torques as depicted, please determine the maximum shearing stress in (a) shaft \(EF\) (radius \(c_{EF} = 25\) mm), (b) shaft \(FG\) (radius \(c_{FG} = 23\) mm), and (c) shaft \(GH\) (radius \(c_{GH} = 21\) mm)

\[ T_F = 1.6 \text{ kN}\cdot\text{m} \quad T_G = 0.8 \text{ kN}\cdot\text{m} \quad T_H = 0.4 \text{ kN}\cdot\text{m} \]
A component with both solid rod EF and hollow tube with flange GH welded onto a flat end plate of GFI, and the rod EF and tube GH share the same longitude axis, as illustrated. The hollow tube GH is fixed at rigid flange of AA’BB’ at H using bolts. The solid rod EF has a diameter $d_{EF} = 3.81$ cm and is made of steel with allowable shearing stress of 82.7 MPa. Tube GH is made of copper alloy with OD $d_2 = 7.62$ cm and ID $d_1 = 6.35$ cm with allowable shearing stress of 48.3 MPa. Determine the largest torque $T$ that can be applied at $E$. 
Two solid shafts $GH$ and $IJ$ are connected by gears $K$ and $L$ as illustrated. The radius for gear $K$ is $R_K = 4$ in, while radius for gear $L$ is $R_L = 2.5$ in. Both two shafts of $GH$ and $IJ$ are made of same steel with allowable shearing stress of 8 ksi. The radius for the shafts are $c_{GH} = 0.8$ in and $c_{IJ} = 0.625$ in. Please determine the largest torque that can be applied at $H$. 
Motor applies torque $T = 500 \text{ N} \cdot \text{m}$ on the HGFE shaft when it is rotating at a constant speed. Knowing shear modulus $G = 27 \text{ GPa}$ and the torques exerted on pulleys B and C are shown. Please calculate the angle of twist between (a) $F$ and $G$, and (b) between $F$ and $H$. Knowing shaft radius $c_{GF} = 2.2 \text{ cm}$; $c_{HG} = 2.4 \text{ cm}$; $c_{EF} = 2.0 \text{ cm}$;
Solid aluminum rod \((G_{Al}=26 \text{ GPa})\) is bonded to solid copper alloy rod \((G_{Al}=39 \text{ GPa})\). The radius for both rods are 10 mm. Calculate the angle of twist with respect to base \(D\) at \(E\) point and at \(F\) point, respectively.

\[
\begin{align*}
D & \quad \text{copper} \quad E & \quad \text{aluminum} \quad F \\
0.2 \text{ m} & \quad & 0.4 \text{ m} & \quad T_F = 50 \text{ N} \cdot \text{m}
\end{align*}
\]
Homework 3.6

A 6 foot long composite shaft consists of 0.2 inch thick copper shell ($G_{Cu} = 5.6 \times 10^6 \text{ psi}$) bonded to 1.2 inch ($G_{Steel} = 11.2 \times 10^6 \text{ psi}$) diameter iron core. If the shaft is subject to 5000 lb•in torque. Please calculate the maximum shearing stress in the steel core and the angle of twist of one end versus the other.
Homework 3.7

Determine maximum shearing stress in a solid shaft of 10 mm diameter as it transmits 2.4 kW at a frequency of (a) 30 Hz and (b) 60 Hz.
Two solid shafts $EF$ and $IJ$ and gears $G (R_G = 3$ inch$)$ and $H (R_H = 5$ inch$)$ are used to transmit 16 hp ($1 \text{ hp} = 6600 \text{ lb\cdotin}$) from motor at $E$ operating at 1200 rpm ($f = 20 \text{ Hz}$) to machine tool at $J$. Knowing the maximum allowable shearing stress is 7.5 ksi. Please calculate the required radius for shaft $EF$ and shaft $IJ$, respectively.