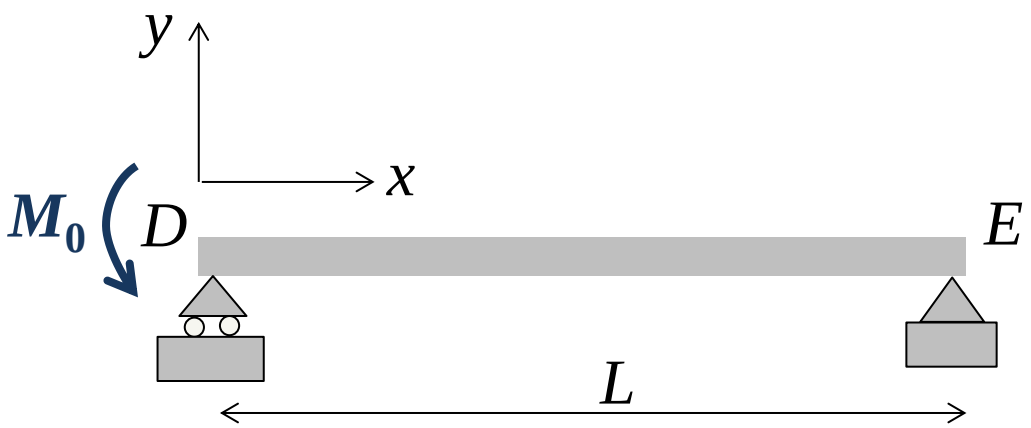
EMA 3702
Mechanics & Materials Science
(Mechanics of Materials)

Chapter 9 Deflection of Beams
Homework Answers/Hints

FLORIDA INTERNATIONAL UNIVERSITY

Homework 9.1

For a simply supported prismatic beam *DE* with local moment at *D*, please step-by-step determine (a) the equation of the elastic curve, (b) the maximum deflection, (c) the slope at the two ends.



Homework 9.1

Draw the FBD for the entire DE beam,

Consider balance of force

???

Consider balance of moment of *D*

???

Therefore, R_E = *??*

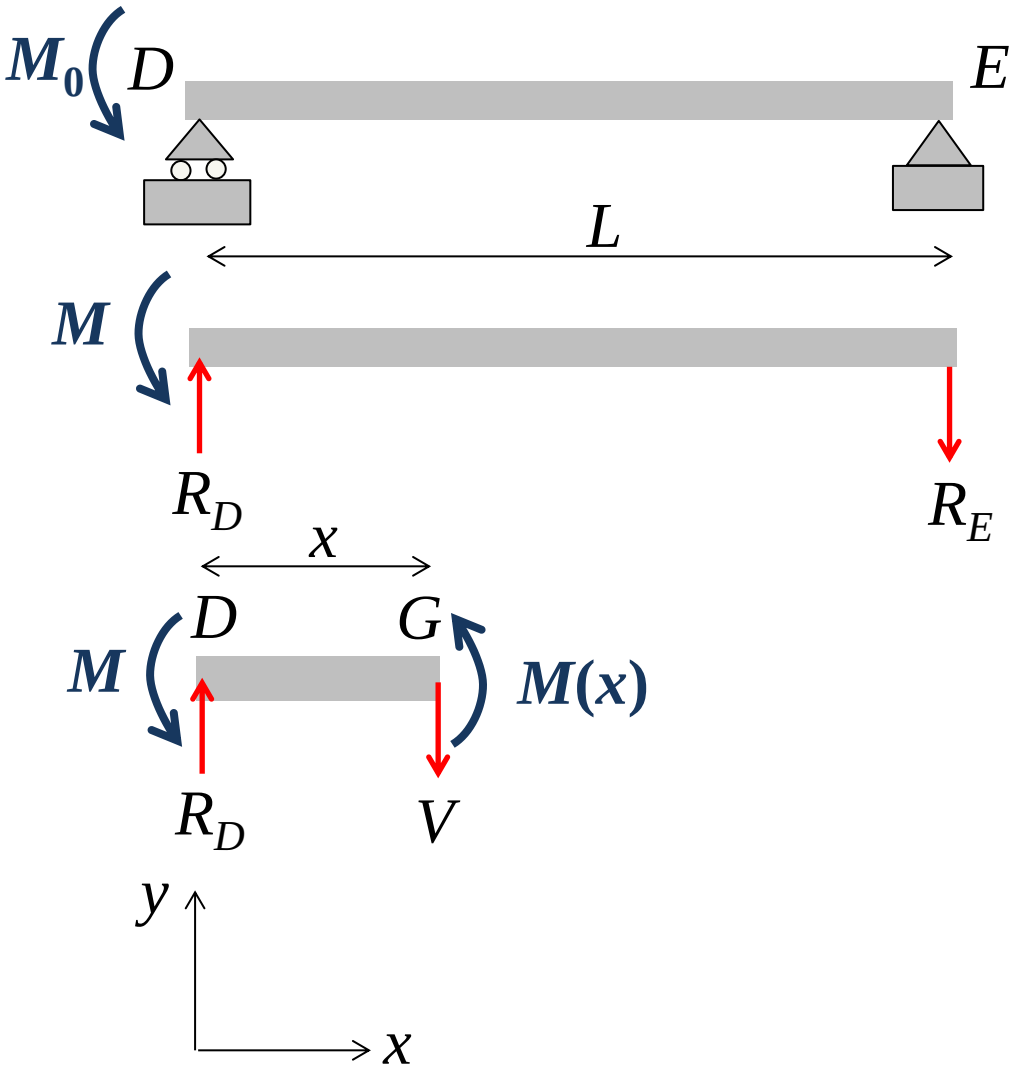
For arbitrary section *DG* from *D*
Balance of force gives

V = *??*

Balance of moment around *D* gives

?? = ??

Therefore, *M* (*x*) = ??



Homework 9.1

From previous, $M(x) = ??$

Therefore, elastic curve satisfies

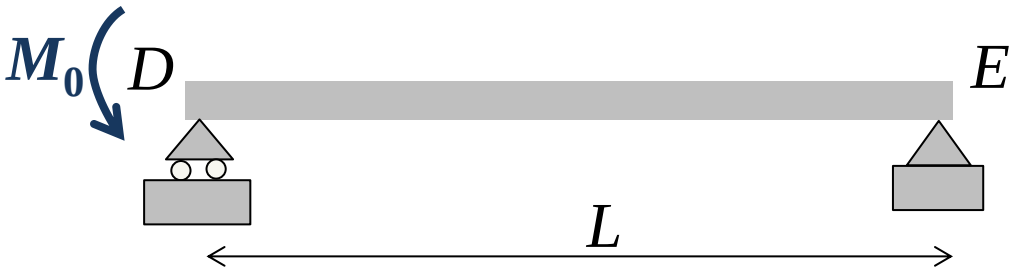
$$EI\frac{d^2y}{dx^2} = M(x) = ??$$

Integrate for once wrt. x

$$EI\frac{dy}{dx} = ?? + C_1$$

Integrate again wrt. x

$$EIy = ?? + C_1x + C_2$$



Consider boundary conditions,

$x = ?, y = ? \rightarrow C_2 = ?;\ x = ?, y = ? \rightarrow\ ?? = ? \rightarrow\ C_1 = ???$

Therefore, elastic curve

$$y = \frac{M_0}{6EIL}x^3 - \frac{M_0}{2EI}x^2 + \frac{M_0L}{3EI}x$$

Homework 9.1

Elastic curve: $$y = \frac{M_0}{6EIL}x^3 - \frac{M_0}{2EI}x^2 + \frac{M_0 L}{3EI}x$$

Maximum deflection occurs when

$$EI\frac{dy}{dx} = ??? = 0$$

Therefore, the meaningful solution will be $$x = \left(1 - \sqrt{\frac{1}{3}}\right)L = 0.423L$$

The corresponding maximum deflection will be

$$y_{max} = \frac{M_0}{6EIL}x^3 - \frac{M_0}{2EI}x^2 + \frac{M_0 L}{3EI}x$$

$$y_{max} = 0.0642 \cdot \frac{M_0 L^2}{EI}$$

Homework 9.1

At $x = 0$

$$EI\frac{dy}{dx} = ??$$

Local slope is

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{M_0 L}{3EI}$$

At $x = L$

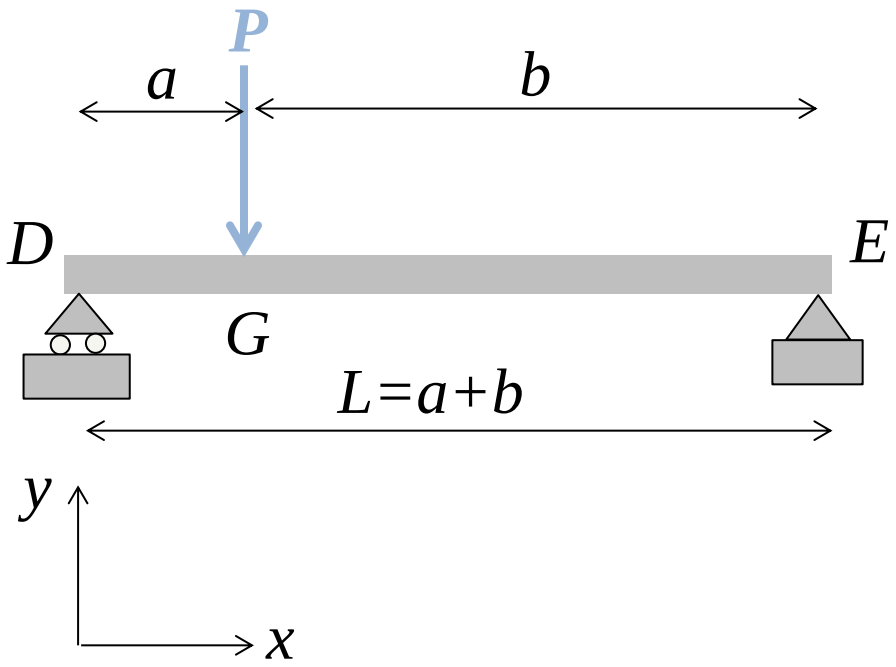
$$EI\frac{dy}{dx} = ??$$

Local slope is

$$\left.\frac{dy}{dx}\right|_{x=L} = -\frac{M_0 L}{6EI}$$

Homework 9.2

For a simply supported prismatic beam DE with concentrated load at *G*, please step-by-step determine (a) the elastic curve *y*(*x*) for 0 < *x* < *a* and (b) the deflection at point *G*.



Homework 9.2

Draw the FBD for the entire beam,
Consider the balance of force

$?? = ?$

Consider the balance of moment around G

$? = ?$

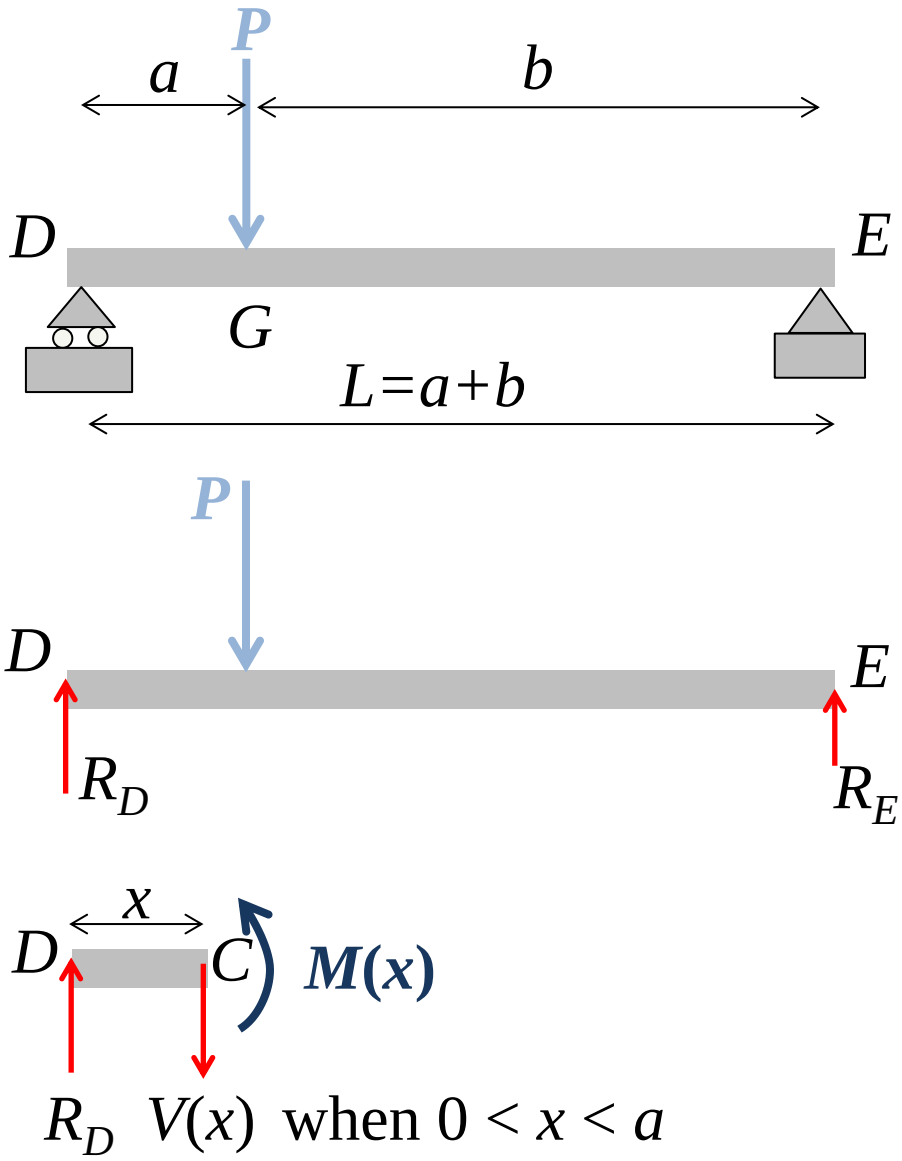
$R_D = ??$

$R_E = ??$

Consider arbitrary section DC to the left of G, or $0 < x < a$, draw FBD
Balance of force gives $V = ??$

Balance of moment at C gives

$$M(x) = ??$$



Homework 9.2

From previous, when $0 < x < a$

$M(x) = ?$

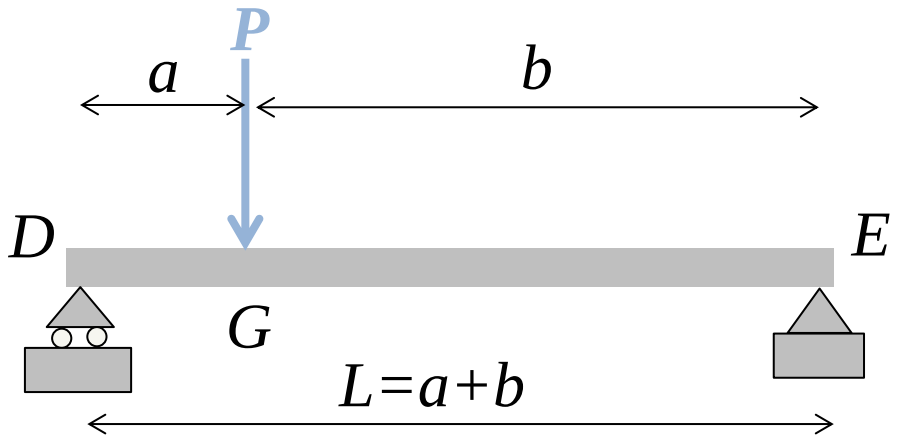
$$EI\frac{d^2y}{dx^2} = M(x) = ?$$

Integrate wrt. x

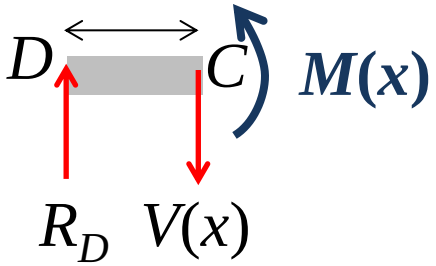
$$EI\frac{dy}{dx} = ? + C_1$$

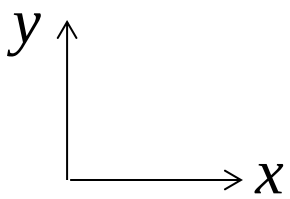
Integrate wrt. x again

$$EIy = ?? + C_2$$



when $0 < x < a$

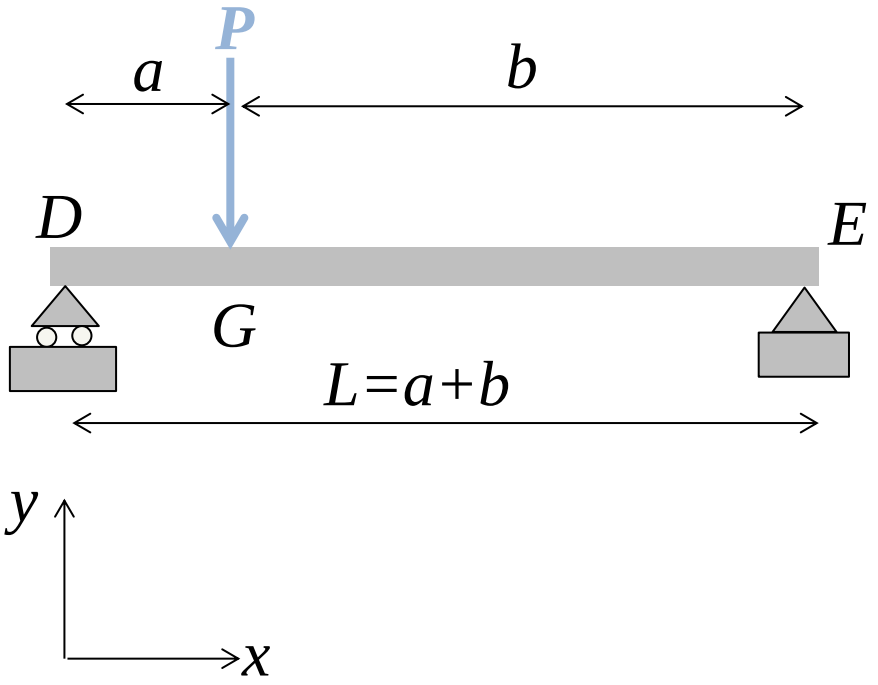




Homework 9.2

Similarly, when $a < x < a+b$, for section CE consider balance of moment around C

$$M(x) = ?$$



Therefore, elastic curve satisfies

$$EI\frac{d^2y}{dx^2} = M(x) = ??$$

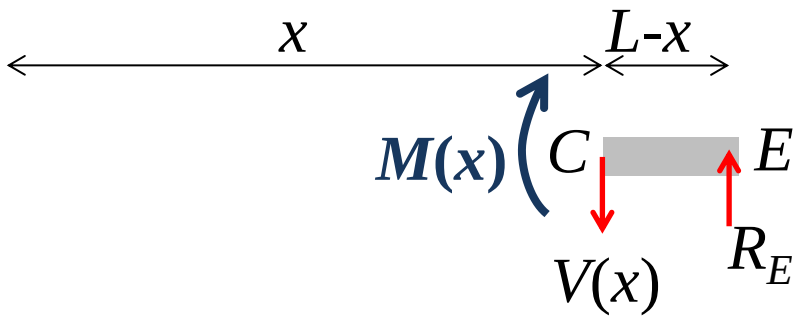
Integrate wrt x

$$EI\frac{dy}{dx} = ?? + C_3$$

Integrate wrt x again,

$$EIy = ?? + C_3x + C_4$$

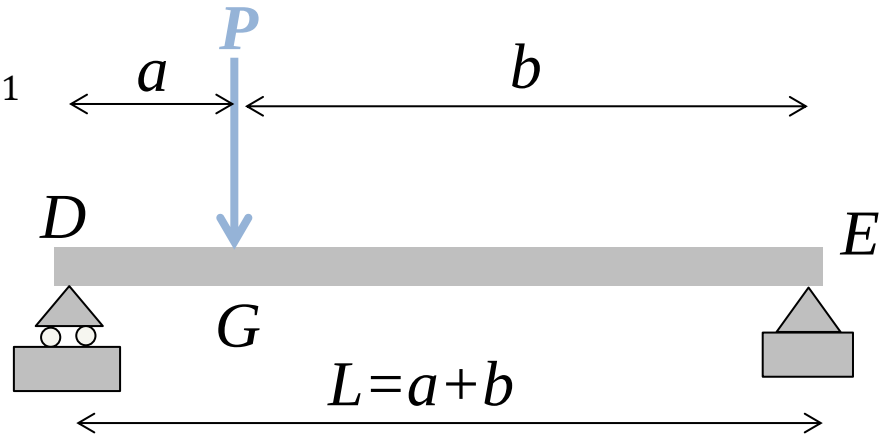
When $a < x < a+b$



Homework 9.2

When $0 < x < a$ $\quad EI\frac{dy}{dx} = \frac{bP}{2(a+b)}x^2 + C_1$

$$EIy = \frac{bP}{6(a+b)}x^3 + C_1x + C_2$$



When $a < x < a+b$

$$EI\frac{dy}{dx} = -\frac{aP}{2(a+b)}x^2 + aPx + C_3$$

$$EIy = -\frac{aP}{6(a+b)}x^3 + \frac{1}{2}aPx^2 + C_3x + C_4$$

Consider boundary conditions

$x = ?, y = ? \rightarrow C_2 = ?$

$x = a + b, y = 0 \rightarrow \quad -\frac{aP}{6(a+b)}(a+b)^3 + \frac{1}{2}aP(a+b)^2 + C_3(a+b) + C_4 = 0$

$$\frac{aP}{3}(a+b)^2 + C_3(a+b) + C_4 = 0 \qquad (a+b)C_3 + C_4 = -\frac{aP}{3}(a+b)^2$$

Homework 9.2

Consider the boundary condition that deflection has to match where the load P is

$x = a, y_{Left} = y_{R,ight}$

Therefore

$$EIy = \frac{bP}{6(a+b)}a^3 + C_1\mathrm{a} = -\frac{aP}{6(a+b)}a^3 + \frac{1}{2}Pa^3 + C_3a + C_4$$

$$aC_1 - aC_3 - C_4 = -\frac{aP}{6(a+b)}a^3 - \frac{bP}{6(a+b)}a^3 + \frac{1}{2}Pa^3$$

$$aC_1 - aC_3 - C_4 = -\frac{a+b}{6(a+b)}Pa^3 + \frac{1}{2}Pa^3 = -\frac{1}{6}Pa^3 + \frac{1}{2}Pa^3$$

$$aC_1 - aC_3 - C_4 = \frac{1}{3}Pa^3$$

Homework 9.2

Consider the other boundary condition that the slope for the elastic curve has to match where the load P is

$x = a,\ \theta_{Left} = \theta_{Right}$

$$EI\frac{dy}{dx} = \frac{bP}{2(a+b)}a^2 + C_1 = -\frac{aP}{2(a+b)}a^2 + aPa + C_3$$

$$C_1 - C_3 = -\frac{aP}{2(a+b)}a^2 - \frac{bP}{2(a+b)}a^2 + Pa^2$$

$$C_1 - C_3 = -\frac{(a+b)}{2(a+b)}Pa^2 + Pa^2$$

$$C_1 - C_3 = \frac{1}{2}Pa^2$$

Homework 9.2

Therefore,

$$C_1 - C_3 = \frac{1}{2}Pa^2 \Rightarrow aC_1 - aC_3 = \frac{1}{2}Pa^3$$

$$aC_1 - aC_3 - C_4 = \frac{1}{3}Pa^3$$

$$\frac{1}{2}Pa^3 - C_4 = \frac{1}{3}Pa^3 \Rightarrow C_4 = \frac{1}{6}Pa^3$$

$$(a+b)C_3 + C_4 = -\frac{aP}{3}(a+b)^2$$

$$(a+b)C_3 + \frac{1}{6}Pa^3 = -\frac{aP}{3}(a+b)^2$$

$$(a+b)C_3 = -\frac{1}{6}Pa^3 - \frac{aP}{3}(a+b)^2 \Rightarrow C_3 = -\frac{1}{6(a+b)}Pa^3 - \frac{aP}{3}(a+b)$$

$$C_1 = C_3 + \frac{1}{2}Pa^2 = -\frac{1}{6(a+b)}Pa^3 - \frac{aP}{3}(a+b) + \frac{1}{2}Pa^2$$

$$C_1 = -\frac{1}{6(a+b)}Pa^3 - \frac{Pab}{3} + \frac{1}{6}Pa^2$$

Homework 9.2

Therefore, the deflection curve for $0 < x < a$ will be

$$EIy = \frac{bP}{6(a+b)EI}x^3 + C_1 x$$

$$C_1 = -\frac{1}{6(a+b)}Pa^3 - \frac{Pab}{3} + \frac{1}{6}Pa^2$$

$$y = \frac{1}{EI}\left\{\frac{bP}{6(a+b)}x^3 + [-\frac{1}{6(a+b)}Pa^3 - \frac{Pab}{3} + \frac{1}{6}Pa^2]x\right\}$$

$$y = \frac{1}{EI}\left\{\frac{P}{6(a+b)}bx^3 + [-\frac{Pa^3}{6(a+b)} - \frac{Pab\cdot 2(a+b)}{6(a+b)} + \frac{Pa^2\cdot(a+b)}{6(a+b)}]x\right\}$$

$$y = \frac{P}{6EI(a+b)}\{bx^3 + [-a^3 - 2a^2b - 2ab^2 + a^3 + a^2b]x\}$$

$$y = \frac{P}{6EI(a+b)}\{bx^3 + [-a^2b - 2ab^2]x\}$$

$$y = \frac{bP}{6EI(a+b)}[x^3 - (a^2 + 2ab)\,x]$$

Homework 9.2

Therefore, the deflection curve for $0 < x < a$ will be

$$y = \frac{bP}{6EI(a+b)}[x^3 - (a^2 + 2ab)\,x]$$

When $x = a$, the y deflection is

$$y = \frac{bP}{6EI(a+b)}[a^3 - (a^2 + 2ab)\,a]$$

$$y = \frac{bP}{6EI(a+b)}[a^3 - a^3 - 2a^2b]$$

$$y = -\frac{2a^2b^2P}{6EI(a+b)}$$

$$y = -\frac{Pa^2b^2}{3EI(a+b)}$$

Homework 9.3

For a prismatic cantilever beam *DE* with *D* side on roller and subject to moment M_0 and *E* side fixed, please step-by-step determine (a) the elastic curve *y*(*x*) and (b) reactions at *D* and *E*

Draw FBD for the entire beam

Balance of force gives:

? = ?

Balance of moment around *E* gives

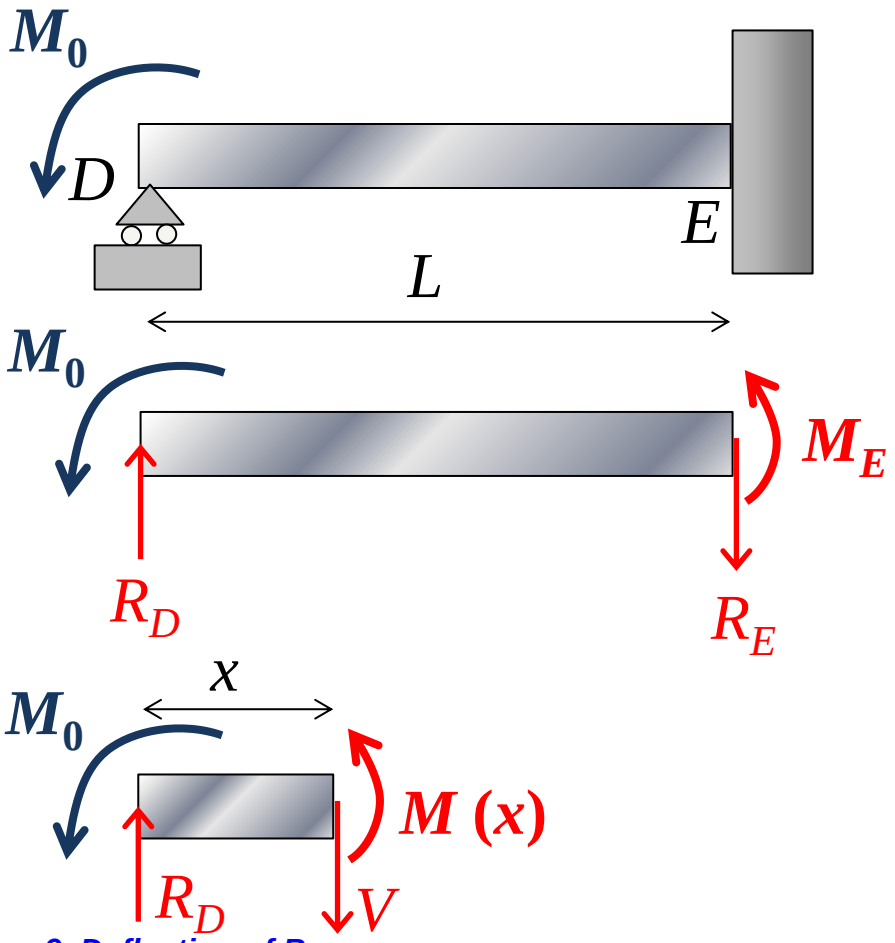
? = ?

Three variables and two equations → a statically indeterminate problem!

Consider arbitrary section *DG*, draw FBD

Consider balance of moment around G

M (*x*) = ??

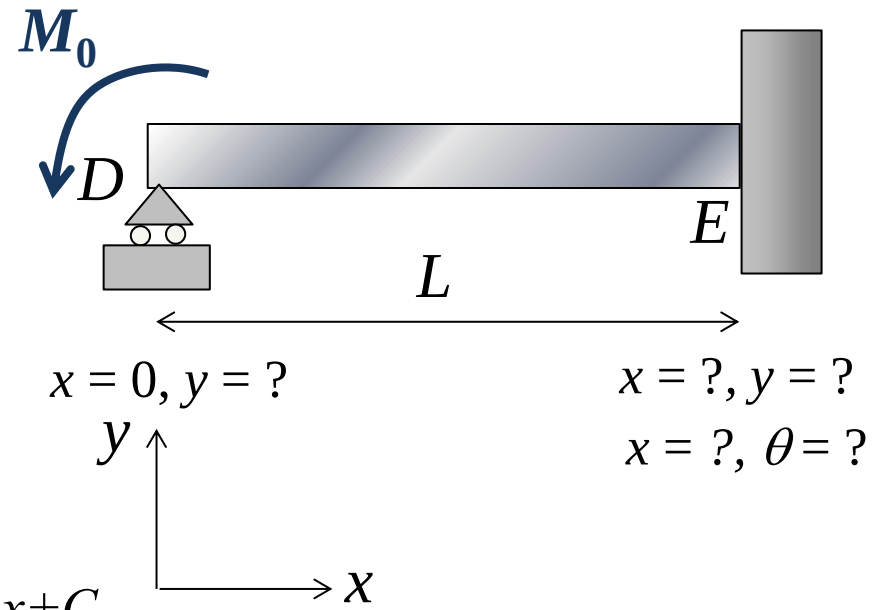


Homework 9.3

Internal bending moment $M(x)$ = ??

The elastic curve satisfies

$$EI\frac{d^2y}{dx^2} = M(x) = ?$$



Integrate wrt. x

$$EI\frac{dy}{dx} = ? + C_1$$

Integrate wrt. x again

$$EIy = \frac{1}{6}R_D x^3 - \frac{1}{2}M_0 x^2 + C_1 x + C_2$$

Totally (3+2) = 5 variables; Three boundary conditions, totally (2+3) = 5 equations

$x = 0, y = 0 \rightarrow \quad C_2 = 0$

$x = L, y = ? \rightarrow \quad \frac{1}{6}R_D L^3 - \frac{1}{2}M_0 L^2 + C_1 L = ?$

$x = L, \theta = ? \rightarrow \quad \frac{1}{2}R_D L^2 - M_0 L + C_1 = ?$

Homework 9.3

Therefore,

$$R_D = R_E$$

$$M_0 + M_E = R_D L$$

$$\frac{1}{6} R_D L^3 - \frac{1}{2} M_0 L^2 + C_1 L = 0$$

$$\frac{1}{2} R_D L^2 - M_0 L + C_1 = 0 \Rightarrow \frac{1}{2} R_D L^3 - M_0 L^2 + C_1 L = 0$$

$$\frac{1}{6} R_D L^3 - \frac{1}{2} M_0 L^2 = \frac{1}{2} R_D L^3 - M_0 L^2$$

$$\Rightarrow \frac{1}{3} R_D L^3 = \frac{1}{2} M_0 L^2 \Rightarrow R_D = \frac{3M_0}{2L}$$

$$R_D = R_E$$

$$R_E = \frac{3M_0}{2L}$$

$$M_0 + M_E = R_D L \Rightarrow M_E = RDL - M_0 = \frac{3M_0}{2L} L - M_0 \Rightarrow M_E = \frac{M_0}{2}$$

$$\frac{1}{2} R_D L^2 - M_0 L + C_1 = 0 \Rightarrow C_1 = M_0 L - \frac{1}{2} R_D L^2 = M_0 L - \frac{1}{2} \cdot \frac{3M_0}{2L} L^2 \Rightarrow C_1 = \frac{M_0 L}{4}$$

Homework 9.3

The elastic curve satisfy

$$EIy = \frac{1}{6}R_D x^3 - \frac{1}{2}M_0 x^2 + C_1 x$$

$$C_1 = \frac{M_0 L}{4} \qquad R_D = \frac{3M_0}{2L}$$

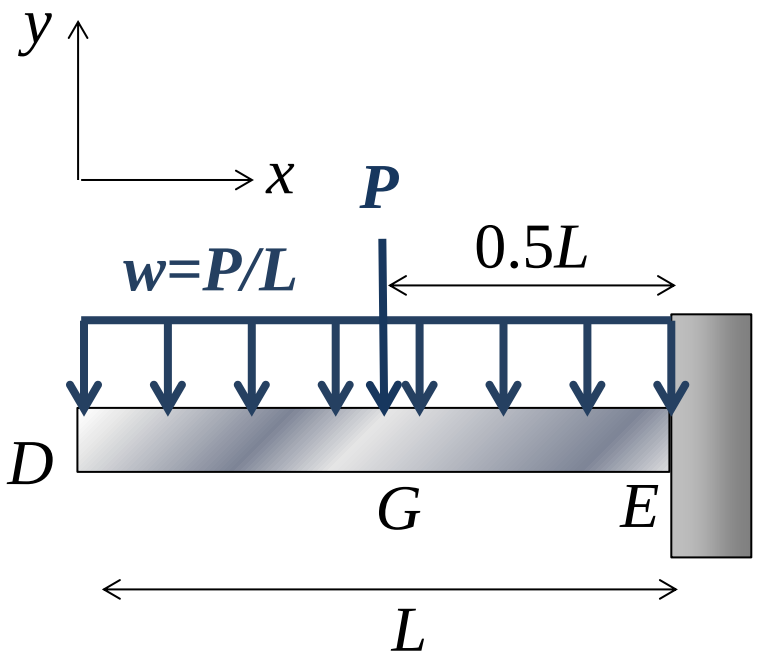
$$EIy = \frac{1}{6}\cdot\frac{3M_0}{2L}x^3 - \frac{1}{2}M_0 x^2 + \frac{M_0 L}{4}x$$

$$EIy = \frac{M_0}{4L}x^3 - \frac{1}{2}M_0 x^2 + \frac{M_0 L}{4}x$$

$$y = \frac{M_0}{4EIL}x^3 - \frac{1}{2EI}M_0 x^2 + \frac{M_0 L}{4EI}x$$

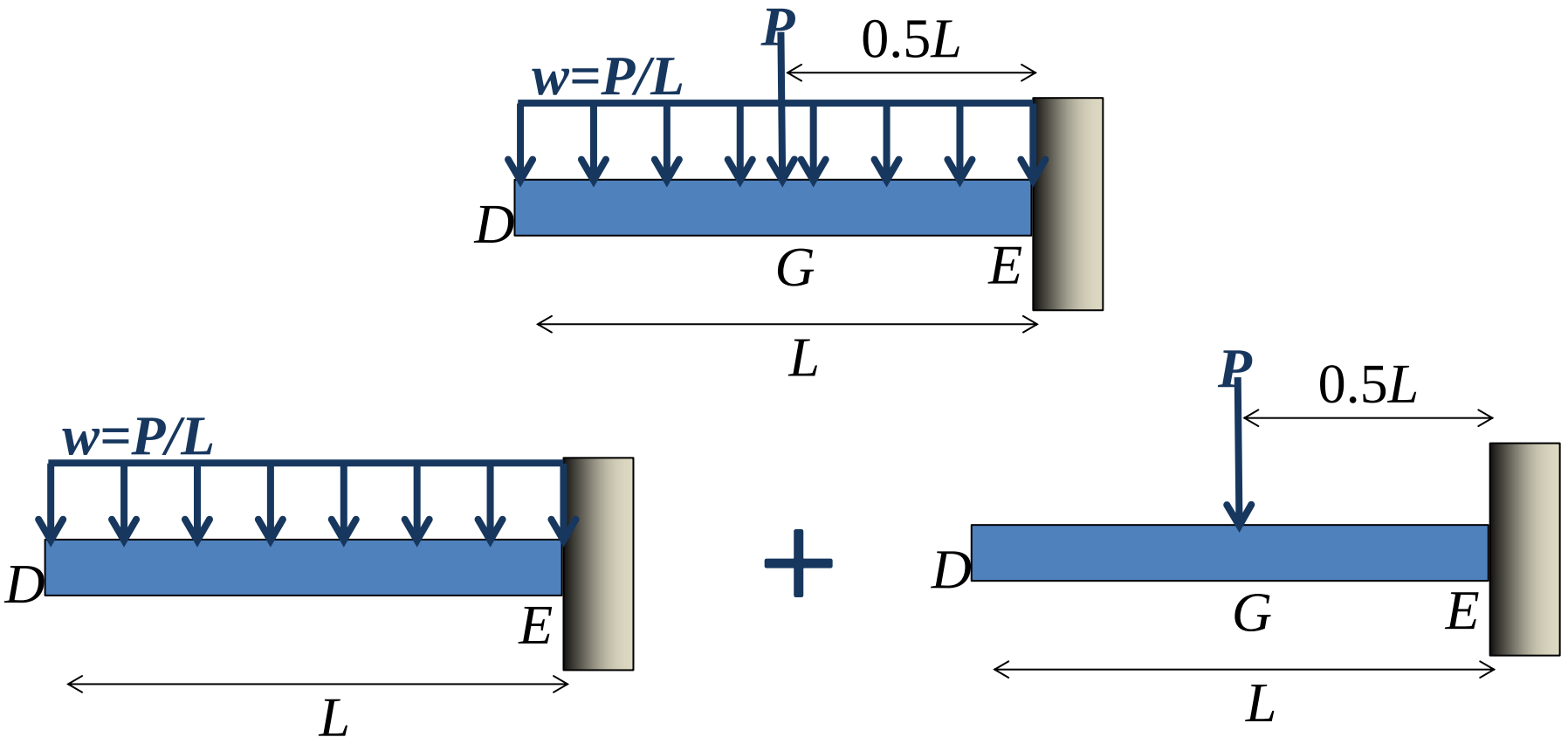
Homework 9.4

For cantilever beam with both distributed and concentrated load as illustrated, please determine the slope and deflection at *D* end. (Hint: to treat concentrated load at G *alone*, section *GE* will behave like a cantilever beam under vertical load at the end (case 1 of textbook Appendix D), while section *DG* remains straight.)



Homework 9.4

The stress states is superposition of the following two



$$(y_D) = (y_D)_w + (y_D)_P$$

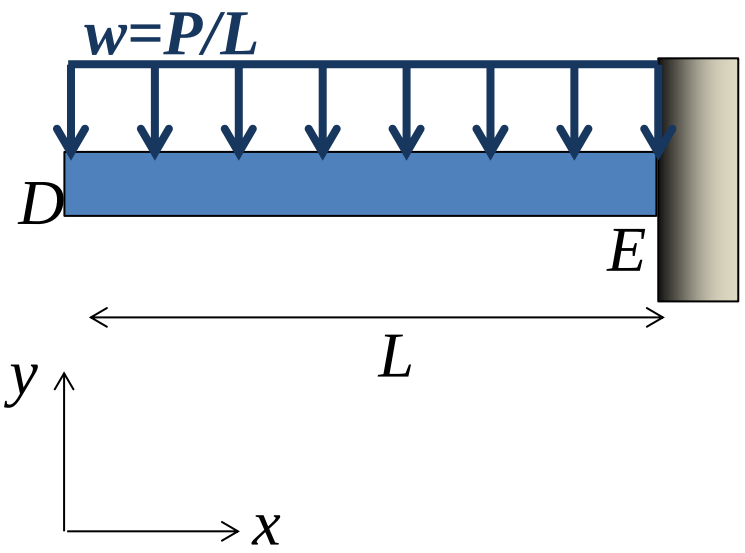
$$\theta_D = (\theta_D)_w + (\theta_D)_P$$

Homework 9.4

For distributed load *w*, from appendix D case 2

Deflection at *D* due to distributed load *w=P/L*

$$(y_D)_w = ?$$



Slope at end *D* due to due to distributed load *w=P/L*

$$(\theta_D)_w = ?$$

Note that **positive slope** instead of negative slope is used here due to the origin of the coordinate system is chosen at *D*, while *x* axis is long the direction from *D* to *E*

Homework 9.4

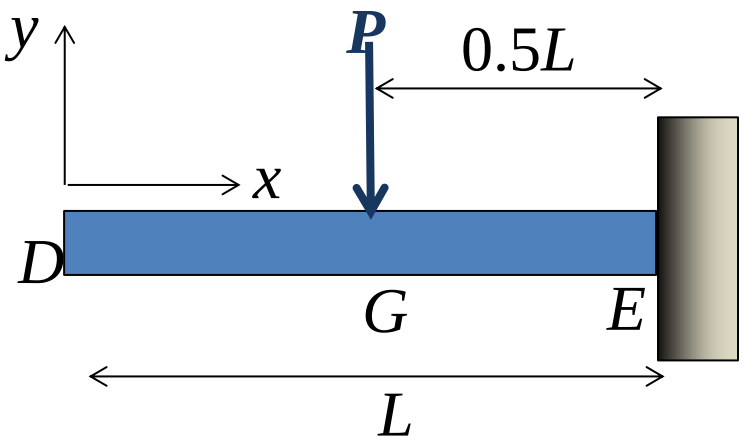
For concentrated load P, from appendix D case 1

Deflection at G due to concentrated load P

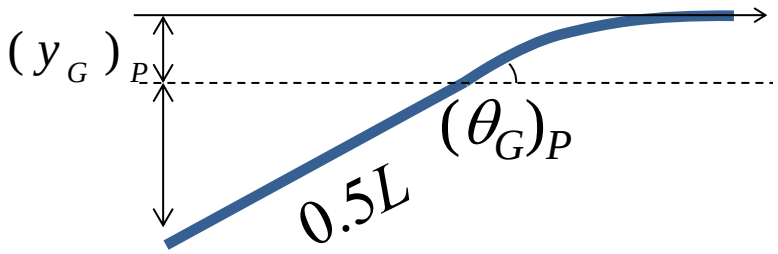
$$(y_G)_P = ? = -\frac{PL^3}{24EI}$$

Slope at G due to concentrated load P

$$(\theta_G)_P = ?$$



Note that **positive slope** instead of negative slope is used here due to the origin of the coordinate system is chosen at D, while x axis is long the direction from D to E



Deflection at G due to concentrated load P

$$(y_D)_P = (y_G)_P - (\theta_G)_P \cdot 0.5L$$

$$(y_D)_P = ?? = -\frac{5PL^3}{48EI}$$

Slope at G due to concentrated load P will be

$$(\theta_D)_P = (\theta_G)_P = ??$$

Homework 9.4

The total deflection at D

$$(y_D) = (y_D)_w + (y_D)_P$$

$(y_D)_w$ = ??

$(y_D)_P$ =??

$$(y_D) = -\frac{PL^3}{8EI} - \frac{5PL^3}{48EI} \Rightarrow \boxed{(y_D) = -\frac{11PL^3}{48EI}}$$

The total slope at D

$$\theta_D = (\theta_D)_w + (\theta_D)_P$$

$(\theta_D)_w$ = ??

$(\theta_D)_P = (\theta_G)_P$ =??

$$(\theta_D) = \frac{PL^2}{6EI} + \frac{PL^2}{8EI} \Rightarrow \boxed{(\theta_D) = \frac{7PL^2}{24EI}}$$